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ON THE STABILITY OF THE STATIONARY FRONT OF AN EXOTHERMIC REACTION IN A CONDENSED PHASE*

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A system in the theory of combustion is considered in which the Arrhenius dependence of the rate of reaction on temperature is modified at temperatures close to the initial temperature in order to ensure the existence of a stationary front. Investigation of the stability of the stationary wave leads to the determination of the neutral curves and the amplitude equation. A coefficient, which defines the nature of the loss of stability, is calculated for the case when there is loss of stability with the occurrence of selfexcited oscillations of the plane front. This coefficient is always negative, which corresponds to soft excitation of selfexcited oscillations. The results obtained are compared with the data from numerical experiments /1-5/.

The heat-conduction equation and the equation for the rate of a chemical equation in a coordinate system which is moving at a velocity U have the form

$$\frac{\partial X}{\partial \tau} = \Delta X + \frac{\partial X}{\partial s} - \Phi, \quad \frac{\partial Y}{\partial \tau} = \frac{\partial Y}{\partial s} - \Phi \tag{1}$$

$$\Phi = pYf(X, C, \theta), \quad f(X, C, \theta) = \exp\left(\frac{1}{\theta}\frac{CX}{CX-1}\right) \tag{2}$$

$$\tau = \frac{U^2}{a}t, \quad s = \frac{U}{a}(z_3 - Ut), \quad x = \frac{U}{a}z_1, \quad y = \frac{U}{a}z_2$$

$$X = \frac{1}{C}\left(1 - \frac{T}{T_1}\right), \quad Y = n, \quad T_1 = T_0 + \frac{Q}{c}, \quad C = \frac{T_1 - T_0}{T_1}$$

$$\theta = \frac{RT_1}{E}, \quad p = \frac{aB}{U^2}\exp\left(-\frac{1}{\theta}\right), \quad \Delta = \Delta_\perp + \frac{\partial^2}{\partial s^2}, \quad \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Here, z_1, z_2 and z_3 are Cartesian coordinates, Φ is the rate of reaction, T and n are the temperature and concentration of the reagent, a and c are the thermal diffusivity and heat capacity, B is a pre-exponential term, T_0 is the initial temperature, T_1 is the combustion temperature, Q is the heat of reaction, E is the energy of activation, R is the gas constant and C, θ and p are dimensionless parameters of the problem.

We shall modify the Arrhenius dependence (2) in the neighbourhood of $X = 1$ by putting it identically equal to zero when $X > X_*$ in order to ensure the existence of a stationary combustion wave. For this purpose, we shall solve the equation with $X' = CX$ and $Y' = CY$

$$\frac{dX'}{ds} = Y' - X', \quad \frac{dY'}{ds} = pY'\exp\left(\frac{1}{\theta}\frac{X'}{X'-1}\right) \tag{3}$$

$$s \to -\infty, \quad X' = \exp(ps), \quad Y' = (1+p)\exp(ps)$$

up to $s = s_*(p, \theta)$, where the magnitude of dY'/dX' is a minimum, and determine $C(p, \theta) = Y'(s_*)$ and $X_*(p, \theta) = X'(s_*)/C$. The dependence of p on the parameters of the problem, C and θ, is thereby determined and this also means the velocity U of the stationary wave. The choice of the discontinuous modification of f is fixed by the definition of X_*.

p and θ are subsequently taken as the independent parameters of the problem. The stationary solution of (1) with the modified dependence (2) is now determined, when $s < s_*$, by the soluton of (3): $X_0 = X'/C$, $Y_0 = Y'/C$ and, when $s > s_*$, we obtain

$$Y_0 = 1, \quad X_0 = 1 + (X_0(s_*) - 1)\exp(s_* - s)$$

The stationary solution which has been found satisfies the conditions

$$s = -\infty, \quad X_0 = Y_0 = 0, \quad s = +\infty, \quad X_0 = Y_0 = 1 \tag{4}$$

A second dependence $C = C(p, \theta)$ is defined when a second modification of f is chosen from the condition for the existence of a stationary solution of (1) which satisfies (4).

In order to determine $C(p, \theta)$ and the neutral curves, it is sufficient to use the

*Prikl. Matem. Mekhan., 54, 6, 978-987, 1990 discontinuous modification of f which has been obtained (compare with /6/), but the requirements regarding the smoothness of the modification of f need to be increased for the correct determination of the Lyapunov coefficients. Calculations show that the dependence $C(p, \theta)$, the neutral curves, the Lyapunov coefficient $G_R(\theta)$ and the other characteristics depend only slightly on the modifications obtained by a change in the discontinuous modification which has been constructed in the neighbourhood of $X = X_*$.

The initial-boundary value problem (Ω is a cylinder and ν is the external normal to the boundary of the cylinder)

$$X|_{\tau=0} = X^0(x, y, s), \quad Y|_{\tau=0} = Y^0(x, y, s), \quad \left.\frac{\partial X}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial Y}{\partial \nu}\right|_{\partial\Omega} = 0 \tag{5}$$

is formulated in the case of Eq.(1).

In order to study the neighbourhood of the stationary solution, we shall reduce problem (1), (5) to the local form /7/

$$\frac{\partial \Psi}{\partial \tau} = A\Psi + \frac{1}{2!}B(\Psi, \Psi) + \frac{1}{3!}C(\Psi, \Psi, \Psi) + \ldots \tag{6}$$
$$\Psi|_{\tau=0} = \Psi^0, \quad \partial\Psi/\partial\nu|_{\partial\Omega} = 0$$

Here,

$$\Psi = [U, \ V]^T, \quad U = X - X_0, \quad V = Y - Y_0, \quad \Psi_n = [U_n, \ V_n]^T, \ n = 1, \ 2, \ 3$$
$$A = \begin{Vmatrix} \Delta + \partial/\partial s - \Phi_X & -\Phi_Y \\ -\Phi_X & \partial/\partial s - \Phi_Y \end{Vmatrix}$$
$$B(\Psi_1, \ \Psi_2) = -[D^2\Phi, \ D^2\Phi]^T, \ C(\Psi_1, \ \Psi_2, \ \Psi_3) = -[D^3\Phi, \ D^3\Phi]^T$$
$$D^2\Phi(\Psi_1, \ \Psi_2) = \Phi_{XX}U_1U_2 + \Phi_{XY}(U_1V_2 + V_1U_2) + \Phi_{YY}V_1V_2$$
$$D^3\Phi(\Psi_1, \ \Psi_2, \ \Psi_3) = \Phi_{XXX}\ U_1U_2U_3 + \Phi_{YYY}\ V_1V_2V_3 + \Phi_{XXY}(U_1U_2V_3 + U_1V_2U_3 + V_1U_2U_3) + \Phi_{XYY}(U_1V_2V_3 + V_1U_2V_3 + V_1V_2U_3)$$

The derivatives of Φ with respect to X and Y are calculated for the stationary solution X_0, Y_0.

In order to study the stability of the state $\Psi = 0$ of Eq.(6), we obtain the eigenvalue problem

$$A\psi = \lambda\psi, \quad \psi \to 0, \quad s \to \pm\infty, \quad \partial\psi/\partial\nu|_{\partial\Omega} = 0 \tag{7}$$

($\psi = [u, v]^T$ depends on x, y and s). This problem is solved in the same manner as in /6/. In solution, which corresponds to $\lambda = 0$, always exists: $\psi_0 = [dX_0/ds, \ dY_0/ds]^T$.

Let us now apply the method of separation of variables to problem (7)

$$\psi(x, \ y, \ s) = \varphi(s)\, w(x, \ y), \quad \varphi = [\xi, \ \eta]^T \tag{8}$$

and obtain the two problems

$$d^2\xi/ds^2 + d\xi/ds - (\Phi_X + \lambda + \mu)\,\xi - \Phi_Y\eta = 0 \tag{9}$$
$$d\eta/ds - \Phi_X\xi - (\Phi_Y + \lambda)\,\eta = 0; \quad \xi, \ \eta \to 0, \quad s \to \pm\infty$$
$$\Delta_\perp w + \mu w = 0, \quad \partial w/\partial\nu|_{\partial\Omega} = 0 \tag{10}$$

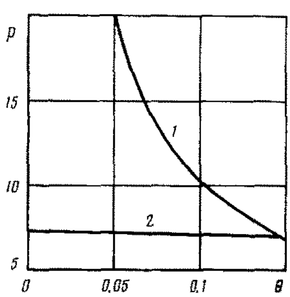

Fig.1

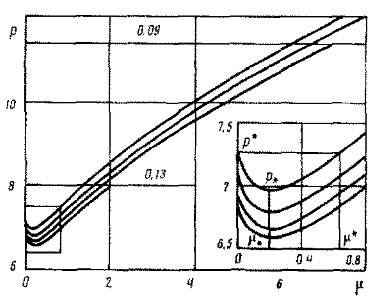

Fig.2

In order to solve problem (9), which depends on an additional parameter $\mu \geqslant 0$, let us define two bases of the solutions of (9): $\varphi^- = (\varphi_1^-, \varphi_2^-, \varphi_3^-)$ and $\varphi^+ = (\varphi_1^+, \varphi_2^+, \varphi_3^+)$ which are specified by their own asymptotic forms

$$\begin{gathered} s \to -\infty, \ \varphi_1^- \sim [p, \ (p+\lambda)^2 + p - \mu]^T C^{-1} \exp (p+\lambda) s \\ \varphi_m^- \sim [1, \ 0]^T C^{-1} \exp (p_m s) \\ s \to +\infty, \ \varphi_1^+ \sim [0, \ 1]^T C^{-1} \exp (\lambda s), \ \varphi_m^+ \sim [1, \ 0]^T C^{-1} \exp (p_m s) \\ p_{2,3} = -{}^1/_2 \pm ({}^1/_4 + \lambda + \mu)^{1/2} \ (m = 2, \ 3) \end{gathered} \tag{11}$$

The two bases φ^- and φ^+ are linearly dependent and the constant (3×3) matrix $S(\lambda)$ is therefore defined as

$$\varphi_m^- = S_{nm}\varphi_n^+, \ m, \ n = 1, \ 2, \ 3$$

The dispersion equation for λ is determined in terms of its coefficients. In particular, in the case of a λ which belongs to the first quadrant, we obtain /6/

$$\Delta(\lambda, \ p, \ \theta, \ \mu) \equiv S_{11}S_{22} - S_{12}S_{21} = 0 \tag{12}$$

The neutral curve $p = p(\theta, \mu)$ is determined from the condition $\operatorname{Re}\lambda(p, \theta, \mu) = 0$ for the root of (12) with the maximum $\operatorname{Re}\lambda$. If the imaginary part of this root differs from zero, then the function $\omega(\theta, \mu) = \operatorname{Im}\lambda(p, \theta, \mu)$ is defined when $\operatorname{Re}\lambda = 0$. The function $\varphi = S_{22}\varphi_1^- - S_{21}\varphi_2^-$ which corresponds to the λ which has been found determines the eigenfunction ψ in accordance with (8).

The results of the calculations are shown in Figs.1 and 2 and in Table 1. Cross-sections of the neutral surface $p = p(\theta, \mu)$ when $\theta = 0.04N - 0.03$, where $N = 1, \ldots, 4$ are shown in Fig.2. Here, on the neutral curves, the values of p decrease as the number N increases and, in particular, $p^* = p(\theta, 0)$ also decreases (Table 1). The values of $p_*(\theta)$ correspond to the minimum on the neutral curves which is attained when $\mu = \mu_*(\theta)$. The frequencies $\omega^*(\theta)$ and $\omega_*(\theta)$ are determined using $\omega(\theta, \mu)$ with $\mu = \mu^*$ and $\mu = \mu_*$ respectively.

The boundary of one-dimensional stability $p = p^*(\theta)$ is shown in Fig.1 in the θ, p plane (curve 2). A comparison with the data from /3/, where this curve is given in the form $C = 9.1\theta/(1 + 2.5\theta)$, shows that the relative erorr in the values of p^* does not exceed 0.015. The domain of physical values of the parameters $C(p, \theta) < 1$ is bounded by curve 1 in Fig.1. For fixed θ, the variation in p is bounded by the condition $p < p_{\max}$ which corresponds to $C < 1$ and the neutral curve $p = p(\mu)$, determined for a given θ, therefore divides the strip $0 < p < p_{\max}$ in the μ, p plane into a domain of stability and a domain of instability. The values of $p < p(\mu)$ correspond to the domain of stability. In the case when $\theta = 0.01$ and μ is not small, the neutral curve can be approximated by the parabola $\mu \simeq -0.47 - 0.60p + 1.05p^2$. In Fig.2 the boundary of the physical domain of $p_{\max}$ is indicated for the values $\theta = 0.09$ and $\theta = 0.13$.

Table 1

$\theta \cdot 10^2$	p^*	p_*	$\mu_* \cdot 10^4$	$\mu^* \cdot 10^4$	ω^*	ω_*	$-G_R$
1	7.26	6.97	2012	6389	1.146	1.578	3.0
3	7.16	6.88	2039	6414	1.150	1.582	4.2
5	7.08	6.80	2065	6496	1.158	1.589	5.0
7	7.02	6.73	2118	6608	1.167	1.605	5.7
9	6.96	6.68	2129	6729	1.177	1.614	6.3
11	6.92	6.63	2141	6848	1.186	1.625	6.8
13	6.88	6.58	2202	6960	1.196	1.645	7.1

Estimates of the critical wave number $k_* = \sqrt{\mu_*}$ when $\theta = 0.05$ yield /1/ $k_* \simeq 0.42$ while, according to Table 1, $k_* \simeq 0.45$. According to the data on the period of the oscillations (/1/, $\gamma = 0.128$), $\omega_* \simeq 1.5$ while, from Table 1, $\omega_* \simeq 1.6$. The value $\omega^* \simeq 1.2$ is determined from the period of the one-dimensional oscillations (/3/, Fig.2) for $\theta = 0.0318$ while, in the table, $\omega^* \simeq 1.15$. The calculated values for the velocity of the wave front are therefore also consistent.

Let us now return to a more detailed study of the loss of stability of a stationary wave in the case when auto-oscillations of the plane front occur, and to the calculation of the first Lyapunov coefficient. The loss in stability is associated with the mode $\mu = 0$, $w = 1$ and, in accordance with this, there is no dependence on the x and y coordinates and it is, in fact, a one-dimensional stability problem which is being investigated.

The eigenvalue problem adjoint to (7)

$$A^*\psi^* = \bar{\lambda}\psi^*, \ \partial\psi^*/\partial\nu\,|_{\partial\Omega} = 0 \tag{13}$$

concerning the scalar product

$$\langle\psi_1, \psi_2\rangle = |\Omega|^{-1}\int_\Omega dx\,dy \int \psi_1\cdot\bar{\psi}_2\,ds$$

is next required.

Here

$$A^* = \begin{Vmatrix} \Delta - \partial/\partial s - \Phi_X & -\Phi_X \\ -\Phi_Y & -\partial/\partial s - \Phi_Y \end{Vmatrix}$$

The other boundary conditions for ψ^* require satisfying the condition

$$\langle A\psi_1, \psi_2^*\rangle - \langle\psi_1, A^*\psi_2^*\rangle = 0$$

By applying the method of separation of variables

$$\psi^* = \varphi^*(s)\,w(x, y), \quad \varphi^* = [\xi^*, \eta^*]^T$$

we obtain

$$d^2\xi^*/ds^2 - d\xi^*/ds - (\Phi_X + \bar{\lambda} + \mu)\,\xi^* - \Phi_X\eta^* = 0 \tag{14}$$
$$d\eta^*/ds + \Phi_Y\xi^* + (\Phi_Y + \bar{\lambda})\,\eta^* = 0$$

We shall assume that the eigenvectors (7) and (13) form a bi-orthogonal basis

$$\langle\psi_m, \psi_n^*\rangle = \delta_{mn} \tag{15}$$

In the case under consideration $m, n = 0, 1, 2$ corresponds to $\lambda_0 = 0$ and $\lambda_1 = \lambda$, $\lambda_2 = \bar{\lambda}$ which are associated with neutral perturbations and the occurrence of auto-oscillations. We shall next exclude neutral perturbations $\langle\Psi, \psi_0^*\rangle = 0$. Then, the loss in stability can be considered in the central manifold, the cross-sections of which $\theta = \text{const}$, $p = \text{const}$ are two-dimensional /8/. In the case when Ψ belongs to the central manifold, we define the coordinates a and $\bar{a}$:

$$\Psi = a(\tau)\,\psi + \bar{a}(\tau)\,\bar{\psi} + \Psi', \quad a(\tau) = \langle\Psi, \psi^*\rangle \tag{16}$$

where Ψ' is uniquely defined by the Ψ specified on the central manifold ($\langle\Psi', \psi_0^*\rangle = 0$), and ψ and ψ^* correspond to $\lambda = \lambda_1$.

The contraction of problem (6) into the central manifold leads to a two-dimensional system in a coordinates. Its normal form /8/ has the representation

$$\dot{a} = \lambda a + Ga\,|a|^2 \tag{17}$$

Here, terms of higher order with respect to a have been omitted and a derivative with respect to τ is denoted by a dot. If it is only necessary to determine the basic contribution to the bifurcated periodic solution in powers of $p - p^*$, there is no need to distinguish the normal variables a in (17) and the coordinates in the central manifold (16). Moreover, the contribution from Ψ' in (16) to the bifurcated solution is of the order of $p - p^*$ /9/ and the principal term in the expansion of the bifurcated solution in powers of $p - p^*$, when account is taken of Eq.(17), has the form

$$\Psi = 2\operatorname{Re}\{[-\lambda_R'(p - p^*)/G_R]^{1/2}\,e^{i\omega\tau}\,\psi\} + O(p - p^*) \tag{18}$$

Here λ_R' is the derivative of the real part of λ with respect to p when $p = p^*$ and G_R is the real part of G from (17).

To calculate G_R we shall make use of the formal expansions for the periodic solution (6) with subsequent use of the Fredholm alternative /7/.

The scalar product

$$[\Psi_1, \Psi_2] = \frac{1}{2\pi}\int_0^{2\pi}\langle\Psi_1(t), \Psi_2(t)\rangle\,dt \tag{19}$$

is introduced into the space of functions which are 2π-periodic with respect to t and the operator $D_A = -\omega_0\partial/\partial t + A_0$. Here, $t = \omega_0\tau$, $\omega_0 = \omega^*$ and A_0 is the boundedness of the operator A on the boundary of stability $p = p^*$. The adjoint operator with respect to the scalar product (19) $D_A^* = \omega_0\partial/\partial t + A_0^*$.

The vectors, which are defined on the boundary of stability: $\chi_0 = \psi_0$, $\chi = e^{it}\psi$, $\bar{\chi} = e^{-it}\bar{\psi}$, where ψ_0, ψ and $\bar{\psi}$ are eigenfunctions of the operator A_0, belong to the kernel of the operator D_A. Similarly, $\chi_0^* = \psi_0^*$, $\chi^* = e^{it}\psi^*$, $\bar{\chi}^* = e^{-it}\bar{\psi}^*$ belong to the kernel of D_A^* where ψ_0^*, ψ^* and $\bar{\psi}^*$ are the solutions of the associated problem of the eigenvalues which, together with

ψ_0, ψ and $\bar{\psi}$, form the biorthogonal basis (15).

Let us seek the bifurcated solution of Eq.(6) which is 2π-periodic with respect to $t = \omega(\varepsilon)\tau$ in the form of power series $\varepsilon = [\Psi, \chi^*]$

$$\Psi = \varepsilon\Psi_1(t) + \varepsilon^2\Psi_2(t)/2! + \ldots, \; p(\varepsilon) - p^* = \varepsilon p_1 + \varepsilon^2 p_2/2! + \ldots \tag{20}$$
$$\omega(\varepsilon) - \omega_0 = \varepsilon\omega_1 + \varepsilon^2\omega_2/2! + \ldots$$

In this case, the parameter θ is assumed to be fixed everywhere. In order to exclude neutral perturbations, we require that $[\Psi, \chi_0^*] = 0$. On substituting (20) into (6), we find equations for determining the coefficients in the right-hand sides of (20)

$$D_A\Psi_1 = 0 \tag{21}$$
$$D_A\Psi_2 - 2\omega_1 d\Psi_1/dt + 2p_1 A'\Psi_1 + B(\Psi_1, \Psi_1) = 0 \tag{22}$$
$$D_A\Psi_3 - 3\omega_2 d\Psi_1/dt + 3p_2 A'\Psi_1 - 3\omega_1 d\Psi_2/dt + 3p_1 A'\Psi_2 + \tag{23}$$
$$3p_1 B'(\Psi_1, \Psi_1) + 3p_1^2 A''\Psi_1 + 3B(\Psi_1, \Psi_2) + C(\Psi_1, \Psi_1, \Psi_1) = 0$$

and so on. The operators A, B and C and their derivative with respect to p, which are denoted by a prime, are considered on the boundary of stability $p = p^*$.

The unique solution of Eq.(21), which satisfies the auxiliary conditions $[\Psi_1, \chi_0^*] = 0$, $[\Psi_1, \chi^*] = 1$, is

$$\Psi_1 = \chi + \bar{\chi} \tag{24}$$

The conditions for Eq.(22) to be solvable /7/ lead to the requirement that $\omega_1 = 0$ and $p_1 = 0$ and, in the case of Ψ_2, we get

$$D_A\Psi_2 + B(\Psi_1, \Psi_1) = 0, \; [\Psi_2, \chi_0^*] = [\Psi_2, \chi^*] = 0 \tag{25}$$

From the Fredholm condition for Eq.(23) to be solvable

$$3(-i\omega_2 + p_2\lambda') + 3[B(\Psi_1, \Psi_2), \chi^*] + [C(\Psi_1, \Psi_1, \Psi_1), \chi^*] = 0 \tag{26}$$

we find p_2. By comparing (18) and (20) and using expression (24), we get

$$G_R = {}^1/_2 \operatorname{Re}\{[B(\Psi_1, \Psi_2), \chi^*] + {}^1/_3 [C(\Psi_1, \Psi_1, \Psi_1), \chi^*]\} \tag{27}$$

In order to determine G_R, we find Ψ_2 from Eq.(22)

$$\Psi_2 = e^{2it}Z_2 + Z_0 + e^{-2it}\bar{Z}_2, \; Z_n = [R_n, S_n]^T, \; n = 0, 2$$

Substituting this into (25) we obtain two problems for Z_0 and Z_2

$$d^2R_0/ds^2 + dR_0/ds - \Phi_X R_0 - \Phi_Y S_0 = 2F_0 \tag{28}$$
$$dS_0/ds - \Phi_X R_0 - \Phi_Y S_0 = 2F_0$$
$$F_0 = \Phi_{XX}\xi\bar{\xi} + \Phi_{XY}(\xi\bar{\eta} + \bar{\xi}\eta) + \Phi_{YY}\eta\bar{\eta}$$
$$d^2R_2/ds^2 + dR_2/ds - (\Phi_X + 2i\omega_0)R_2 - \Phi_Y S_2 = F_2$$
$$dS_2/ds - \Phi_X R_2 - (\Phi_Y + 2i\omega_0)S_2 = F_2 \tag{29}$$
$$F_2 = \Phi_{XX}\xi^2 + 2\Phi_{XY}\xi\eta + \Phi_{YY}\eta^2$$

The solution of problem (28) must satisfy the supplementary condition $\langle Z_0, \psi_0^*\rangle = 0$. We obtain

$$G_R = -\operatorname{Re}\left\{\int_{-\infty}^{+\infty}\left(\frac{1}{2!}M_2 + \frac{1}{3!}M_3\right)(\bar{\xi}^* + \bar{\eta}^*)\,ds\right\} \tag{30}$$
$$M_2 = \Phi_{XX}(\xi R_0 + \bar{\xi}R_2) + \Phi_{YY}(\eta S_0 + \bar{\eta}S_2) +$$
$$\Phi_{XY}(\xi S_0 + \bar{\xi}S_2 + \eta R_0 + \bar{\eta}R_2)$$
$$M_3 = 3\Phi_{XXX}\xi^2\bar{\xi} + 3\Phi_{XXY}(2\xi\bar{\xi}\eta + \xi^2\bar{\eta}) +$$
$$3\Phi_{XYY}(2\xi\eta\bar{\eta} + \bar{\xi}\eta^2) + 3\Phi_{YYY}\eta^2\bar{\eta}$$

Here, M_2 and M_3 determine the contribution to G_R from the quadratic part B and cubic part C of Eq.(6). The results of the calculations of G_R and shown in Table 1. Since the magnitude of G_R is negative for all θ, there is always a soft perturbation of the selfexcited oscillations of the plane combustion front on passing across the boundary of stability.

We will point out some details of the calculations. The solutions ψ_0^* and ψ^* of Eqs.(14) corresponding to $\lambda = 0$ and $\lambda = i\omega_0$ are determined by integration with respect to the given asymptotic forms when $s \to -\infty$

$$\varphi_0^* \simeq [(1+p), -p]^T \exp s, \quad \varphi^* \simeq [q_2 + p - i\omega_0, -p]^T \exp (q_2 s)$$
$$q_{1,2} = {}^1/_2 \pm ({}^1/_4 - i\omega_0)^{1/2} \tag{31}$$

At the same time the solution φ_0^* becomes constant as $s \to +\infty$ and $\varphi^* \simeq [c_1 \exp (q_1 s), \ c_2 \exp (i\omega_0 s)]^T$, where c_1 and c_2 are complex constants. Apart from the solution φ_0^* which has been found, there exists for any Φ a further solution $\varphi_0^* = [1, -1]^T$ for which $\langle \varphi_0, \varphi_0^* \rangle = 0$. The calculations showed that $\langle \varphi, \varphi_0^* \rangle = 0$ in the case of the solutions (31) and, in constructing the biorthogonal system (15), there is no need for the second solution φ_0^* and the sole requirements $\langle \varphi_0, \varphi_0^* \rangle = 1$ and $\langle \varphi, \varphi^* \rangle = 1$ can be satisfied by the choice of the coefficients in the solutions (31).

Eqs.(28) have the integral $S_0 = R_0 + dR_0/ds$. Eqs.(28) is integrated from $s = -\infty$, $Z = 0$, and $Z_0 = Z - \langle Z, \psi_0^* \rangle \psi_0$, which satisfies the condition $\langle Z_0, \psi_0^* \rangle = 0$ is then determined using the solution obtained. The magnitude of F_0 as a function of s had a δ'-shaped form.

The unique solution of Eq.(29) was determined in the following manner. For the solutions of this equation the asymptotic forms when $s \to \pm\infty$ are the same as in the case of the solutions of Eq.(9) when $\lambda = 2i\omega_0$, $\mu = 0$. Let us construct the solution Z_2 with the asymptotics forms $s \to -\infty$, $Z_2 \simeq c_1\varphi_1^- + c_2\varphi_2^-$ (see (11)), where $\varphi_1^-, \varphi_2^- \to 0$ when $s \to -\infty$ and c_1 and c_2 are arbitrary complex constants. By using the asymptotics forms (11) for φ_1^+, φ_2^+ and φ_3^+, it is possible to determine the coefficients of the expansion of the solution which has been found $Z_2 = k_1\varphi_1^+ + k_2\varphi_2^+ + k_3\varphi_3^+$. The complex constants k_1, k_2 and k_3 depend on c_1 and c_2. At the calculations show, the constants c_1 and c_2 are uniquely defined by the conditions $k_1 = k_2 = 0$ and it is thereby uniquely defined that $Z_2 \to 0$, $s \to \pm\infty$.

Substitution of the functions which have been found into relationships (30) determines G_R.

Let us now consider some more complex cases of loss of stability. In order to do this, information is required concerning the solutions of problem (10), when Ω is a circle of radius R, a periphery of length l or a square with a side of length l.

In the case of the circle the eigenvalues $\mu_{m,n} = (j_{m,n}/R)^2$, $m = 0, 1, \ldots, n = 1, 2, \ldots$, when $j_{m,n}$ are the roots of the derivatives of the Bessel functions $J_m' (j_{m,n}) = 0$. All the eigenvalues $\mu_{m,n}$ when $m \neq 0$ correspond to double modes (m, n) with orthonormalized eigenfunctions

$$\{w_{m,n}^{(1)}; w_{m,n}^{(2)}\} = C_{m,n} J_m (j_{m,n} r/R) \{\sqrt{2} \cos m\varphi, \sqrt{2} \sin m\varphi\}$$
$$C_{m,n} = j_{m,n}/[(j_{m,n}^2 - m^2)^{1/2} J_m (j_{m,n})], \quad m \neq 0$$

The remaining $\mu_{0,n}$ $(\mu_{0,1} = 0)$ are the single modes $w_{0,n} = J_0 (j_{0,n} r/R)/J_0 (j_{0,n})$.

In the case of the periphery, the eigenvalues $\mu_n = (2\pi n/l)^2$, $n = 1, 2, \ldots$ are double with the eigenfunctions

$$w_n^{(1)} = \sqrt{2} \cos \frac{2\pi n x}{l}, \quad w_n^{(2)} = \sqrt{2} \sin \frac{2\pi n x}{l}, \quad n \neq 0$$

$w_0 = 1$ corresponds to the single eigenvalue $\mu_0 = 0$.

In the case of a square, there are identical eigenvalues among the eigenvalues $\mu_{m,n} = \pi^2 (m^2 + n^2)/l^2$ $(m, n = 0, 1, \ldots)$ and the number of different modes which correspond to a given μ determines the multiplicity of the eigenvalues. The eigenfunctions corresponding to the modes (m, n) are

$$w_{0,} = 1; \quad w_{m,0} = \sqrt{2} \cos \frac{m\pi x}{l}, \quad m = 1, 2, \ldots$$
$$w_{0,n} = \sqrt{2} \cos \frac{n\pi y}{l}, \quad n = 1, 2, \ldots; \quad w_{m,n} = w_{m,0} w_{0,n}$$

We shall subsequently consider case when, apart from $\mu = 0$, there is just a single one or double $\mu \neq 0$ to which one or a double λ respectively with $\lambda_R > 0$ corresponds. We shall denote the amplitudes and eigenfunctions for the single λ by a_1 and $\psi_1 = \varphi w_1$ and for a double λ by a_2 and $\psi_1 = \varphi w_1$ and $\psi_2 = \varphi w_2$, where w_1 and w_2 are the eigenfunctions (10) for the given μ, and φ is the eigenfunction (9) for the given $\lambda(\mu)$. For $\mu = 0$, we denote the amplitude by a_0 and the eigenfunction by $\psi_0 = \varphi_0$ where the latter is the solution of Eq.(9) when $\lambda = \lambda_0$.

The analogue of the representation (16) in the case of a double λ will be

$$\Psi = \sum_{N=0}^{2} (a_N \psi_N + \bar{a}_N \bar{\psi}_N) + \Psi' \tag{32}$$

and, in the case of a single λ,

$$\Psi = \sum_{N=0}^{1} (a_N\psi_N + \bar{a}_N\bar{\psi}_N) + \Psi^c \tag{33}$$

Let us now consider of loss of stability when $\theta = 0.05$ /1, 2/. According to the data in /1/, for the lower and upper stability boundaries we get $p_* = 6.72$ and $p^* = 6.93$ while, according to the data in Table 1, $p_* = 6.800$ and $p^* = 7.084$. In order to collate the numerical experiments /1, 2/ and the results of the solution of Eq.(9), the linear transformation of the values of p calculated from the data in /1, 2/ was carried out which established a match between the upper and lower stability boundaries.

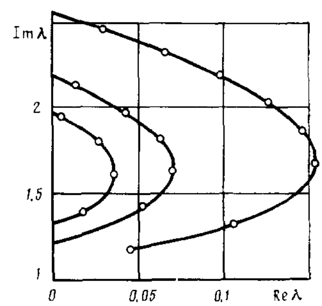

Fig.3

The result of this linear transformation is subsequently indicated as p. The values of λ which correspond to the three different values of p as μ is continuously varied are shown in Fig.3. The discrete values $\mu = 0.1N$ are indicated by the points. $p = 6.895$, $N = 1, \ldots, 4$; $p = 6.989$, $N = 1, \ldots, 5$ are indicated for the values $p_* < p < p^*$. $p = 7.199$, $N = 0, \ldots, 7$ are shown for values of $p > p^*$. Large values of λ_R correspond to large values of p.

For a circular cross-section Ω when $p = 6.895$, $R \simeq 8.7$ (/1/, $\gamma = 0.129$) unstable modes $(2,1)-(1,2)$ correspond to $\mu_2, \ldots, \mu_6$ which are arranged in increasing order. The fundamental $(0,2)$ mode, observed in /1/, with the greatest λ_R, corresponds to a single μ_3 and to the eigenfunction of the fundamental mode $\psi = \varphi w_{0,2}$. If only the fundamental mode were to be unstable then the representation (16) and the amplitude Eq.(17) with the solution $a = \varepsilon e^{i\omega\tau}$ would be valid.

When $p = 6.989$, $R \simeq 8.7$ ([1], $\gamma = 0.128$), the same fundamental mode with a larger number of unstable modes $(1,1)-(2,2)$, which correspond to $\mu_1, \ldots, \mu_5$, was observed.

When $p = 7.199$, $R \simeq 8.6$ ([2], $\gamma = 0.126$) the number of unstable modes $(0,1)-(6,1)$ accompanying the observed fundamental mode was extended: $\mu_0, \ldots, \mu_{10}$.

Let us now consider a simplified case, when only the zeroth and fundamental modes are unstable, in the representation (33) $\psi_0 = \varphi_0$, $\psi_1 = \varphi w_{0,2}$. The normal form of the amplitude equation

$$\begin{aligned} a_0^{\cdot} - \lambda_0 a_0 &= a_0 (G_{00} \mid a_0 \mid^2 + G_{01} \mid a_1 \mid^2) \\ a_1^{\cdot} - \lambda a_1 &= a_1 (G_{10} \mid a_0 \mid^2 + G_{11} \mid a_1 \mid^2) \end{aligned} \tag{34}$$

has the solution $a_0 = 0$, $a_1 = \varepsilon e^{i\omega\tau}$ which is stable when $\operatorname{Re}(\lambda_0 + G_{01}\varepsilon^2) < 0$.

When $p = 7.199$, $R \simeq 4.0$ ([2], $\gamma = 0.126$), the unstable modes $(0,1)-(2,1)$ correspond to μ_0, μ_1 and μ_2. The $(1,1)$ fundamental mode corresponds to a double μ_1.

Let us now consider an example when only the zeroth and fundamental modes are unstable in the representation (32), $\psi_0 = \varphi_0$, $\psi_1 = \varphi w_{1,1}^{(1)}$, $\psi_2 = \varphi w_{1,1}^{(2)}$. The normal form of the amplitude equation

$$\begin{aligned} a_0^{\cdot} - \lambda_0 a_0 &= G a_0 \mid a_0 \mid^2 + H \rho_1 (a)\, a_0 \\ a_1^{\cdot} - \lambda a_1 &= A \rho_1 (a)\, a_1 + iB\rho_2 (a)\, a_2 + D a_1 \mid a_0 \mid^2 \\ a_2^{\cdot} - \lambda a_2 &= -iB\rho_2 (a)\, a_1 + A\rho_1 (a)\, a_2 + D a_2 \mid a_0 \mid^2 \\ (\rho_1 = \mid a_1 \mid^2 &+ \mid a_2 \mid^2,\ \rho_2 = i\, (a_1\bar{a}_2 - \bar{a}_1 a_2)) \end{aligned} \tag{35}$$

was obtained in the following manner. The normal form, which contains all the resonance monomials of order 3, was first determined. Since the initial system (6) permits the group of motions of a circle O (2), the action of this group on the cross-section of the central spectrum leaves the amplitude equations unchanged. The action of the group in the finite dimensional amplitude space induces transformations of the normal amplitudes with regard to which the normal forms of the amplitude equations must be invariant. This leads to constraints on the coefficients of a normal form among which only five independent forms (35) remain. Eq.(35) has a solution in the form of single-helix waves

$$a_0 = 0,\ \{a_1, a_2\} = \varepsilon e^{i\omega\tau} \{1/\sqrt{2}, \pm i/\sqrt{2}\} \tag{36}$$

(the choice of sign corresponds to right-handed and left-handed spirals).

When $p = 7.199$, $R \simeq 6.7$ ([2], $\gamma = 0.126$), the unstable modes correspond to $\mu_0, \ldots, \mu_8$. The fundamental $(2,1)$ mode corresponds to a double μ_2. The solution which has been found has approximately the same form as in the example (32), (36) which had been considered above but has the eigenfunctions $\psi_1 = \varphi w_{2,1}^{(1)}$, $\psi_2 = \varphi w_{2,1}^{(2)}$ which corresponds to a double-helix.

Similar states are observed in the case when the cross-section Ω is a periphery of length l.

When $\theta = 0.0327$, $p \simeq 9.49$, $l \simeq 5.75$ (/4/), the modes μ_0 and μ_1 are unstable. The fundamental mode corresponds to a double μ_1. The representation (32), with $\psi_0 = \varphi_0$, $\psi_1 = \varphi w_1^{(1)}$, $\psi_2 = \varphi w_1^{(2)}$, and Eq.(35) describe the loss in stability as in the case of a circular cross-section Ω. The single-helix wave which is observed corresponds to solution (36).

When $\theta = 0.0327$, $p \simeq 9.49$, $l \simeq 11.5$ (/5/, Fig.1), the modes $\mu_0, \ldots \mu_3$ are unstable. The fundamental mode corresponds to a double μ_1. The observed solution is the same as when $l = 5.75$.

When $\theta = 0.1$, $p \simeq 8.04$, $l \simeq 47.3$ (/5/, Fig.2), the modes $\mu_0, \ldots, \mu_{10}$ are unstable. The fundamental mode corresponds to a double μ_2. The observed solution of the form of (32), (36) with $\psi_1 = \varphi w_2^{(1)}$, $\psi_2 = \varphi w_2^{(2)}$ is a double-helix wave.

Let us now consider cases of loss of stability when $\theta = 0.05$, $p \simeq 7.199$ (/2/, $\gamma = 0.126$) for a square cross-section of various sizes.

When $l \simeq 6.7$, the modes μ_0, μ_1 and μ_2 are unstable. The fundamental (0,1) and (1,0) modes correspond to a double μ_1. The simplest case is when only the zero and fundamental modes are unstable and, in (32), $\psi_0 = \varphi_0, \psi_1 = \varphi w_{0,1}, \psi_2 = \varphi w_{1,0}$. The normal form of the amplitude equation, when account is taken of the symmetry of a square

$$a_0^{\boldsymbol\cdot} - \lambda_0 a_0 = G a_0 \mid a_0 \mid^2 + H (\mid a_1 \mid^2 + \mid a_2 \mid^2) a_0 \tag{37}$$
$$a_1^{\boldsymbol\cdot} - \lambda a_1 = A a_1^2 \bar{a}_1 + B a_2^2 \bar{a}_1 + C a_1 \mid a_2 \mid^2 + D a_1 \mid a_0 \mid^2$$
$$a_2^{\boldsymbol\cdot} - \lambda a_2 = A a_2^2 \bar{a}_2 + B a_1^2 \bar{a}_2 + C a_2 \mid a_1 \mid^2 + D a_2 \mid a_0 \mid^2$$

has a solution of the form of (36) when the lower sign is chosen which corresponds to the observed state.

When $l = 13.4$, the modes $\mu_0, \ldots, \mu_8$ are unstable. The (0,2) and (2,0) fundamental modes correspond to a double μ_3. If just the zero and fundamental modes were to be unstable, the loss of stability could be described by relationships (32), (37) with $\psi_0 = \varphi_0$, $\psi_1 = \varphi w_{0,2}$, $\psi_2 = \varphi w_{2,0}$. In this case, the resulting solution is $a_0 = 0$, $a_1 = a_2 = \varepsilon e^{i\omega\tau}/\sqrt{2}$.

When $l = 9.38$, the modes $\mu_0, \ldots, \mu_4$ are unstable. The fundamental (1,1) mode corresponds to a single μ_2. If only the zero and fundamental modes with $\psi_1 = \varphi w_{1,1}$ are taken into consideration, the loss in stability is described by Eq.(34). The observed state corresponds to the solution $a_0 = 0$, $a_1 = \varepsilon e^{i\omega\tau}$.

In conclusion we note that the determination of the coefficients of the normal forms for the various cases of loss of stability is a fundamental problem. Using the known coefficients, it is possible to solve the problem of the stability of a branching solution.

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